ELLIPTIC GENERA OF SINGULAR VARIETIES

LEV BORISOV AND ANATOLY LIBGOBER

Abstract. Orbifold elliptic genus and elliptic genus of singular varieties are introduced and relation between them is studied. Elliptic genus of singular varieties is given in terms of a resolution of singularities and extends the elliptic genus of Calabi-Yau hypersurfaces in Fano Gorenstein toric varieties introduced earlier. Orbifold elliptic genus is given in terms of the fixed point sets of the action. We show that the generating function for this orbifold elliptic genus $\sum Ell_{orb}(X^n,\Sigma_n)p^n$ for symmetric groups Σ_n acting on n-fold products coincides with the one proposed by Dijkgraaf, Moore, Verlinde and Verlinde. Two notions of elliptic genera are conjectured to coincide.

1. Introduction

This work started as an attempt to understand a beautiful formula for the generating function for the orbifold elliptic genera of symmetric products due to R.Dijkgraaf, G.Moore, E.Verlinde and H.Verlinde, which looks as follows (cf. [17]):

$$\sum_{n\geq 0} p^n Ell_{orb}(X^n/\Sigma_n;y,q)=\prod_{i=1}^{\infty}\prod_{l,m}\frac{1}{(1-p^iy^lq^m)^{c(mi,l)}}. \tag{1.1}$$

Here X is a Kähler manifold, Σ_n is the symmetric group acting on the n-fold product and $c(m,l)$ are the coefficients of the elliptic genus $\sum_{m,l}c(m,l)y^lq^m$ of X. The problem was that the orbifold elliptic genus was defined in physical terms, and the arguments given in [17] did not lend itself to a translation into a mathematical proof.

The two variable elliptic genus is a very compelling invariant for the discussion of which we refer to [12]. Here we just note that it is a holomorphic function on the product of $\mathbf{C}$ and the upper half plane, which is attached to an (almost) complex manifold and is a weak Jacobi form if the manifold is Calabi-Yau. For Calabi-Yau manifolds of a dimension smaller than 12 or equal to 13, the elliptic genus can be expressed in terms of Hirzebruch χ_y genus, but in general, the former contains more information than the latter. In all dimensions elliptic genus specializes into Hirzebruch χ_y genus and in particular into topological Euler characteristic, holomorphic Euler characteristic, signature, etc. Special cases of the formula (1.1) for these invariants have been proved mathematically for some time. For example, it was shown in [26], using the Macdonald formula [30], that if a finite group G acts

Date: to be included.
1991 *Mathematics Subject Classification.* to be included.
Key words and phrases. algebraic geometry, elliptic genera, orbifolds, cobordism, singular varieties, Calabi-Yau varieties.

on a manifold X and

$$e_{orb}(X,G) := \frac{1}{|G|}\sum_{fg=gf} e(X^{f,g}), \tag{1.2}$$

(summation is over all pairs of commuting elements; $X^{f,g}$ is the set of fixed points of both f and g) then:

$$\sum_{n=0}^{n=\infty} e_{orb}(X^n,\Sigma_n) = \prod_i \frac{1}{(1-t^i)^{e(X)}}. \tag{1.3}$$

On the other hand, in [22] (cf. also [19]) it was found that the generating series for the χ_y genera of Hilbert schemes of a surface X is given by:

$$\sum_{n=0}^{n=\infty} \chi_{-y}(X^{[n]})p^n = \exp(\sum_{m=1}^{\infty} \frac{\chi_{-y^m}(X)}{(1-(yp)^m)}\frac{p^m}{m}). \tag{1.4}$$

It was observed in [26], that in the cases when a crepant resolution for X/G does exist, the McKay correspondence (cf. [32]) can be used to prove that the Euler characteristic of such resolution coincides with the orbifold Euler characteristic. This idea was used in a more general case of χ_y genus, with appropriately defined orbifold χ_y genus in [7]. In the case when X is a surface, the Hilbert scheme provides such a resolution (cf. [20]) and hence the left hand side of (1.4) coincides with the generating function for the orbifold χ_y genus of symmetric products of X. Therefore, (1.4) can be viewed as a specialization of (1.1).

This brings in the basic question: how the orbifold Euler characteristic and the orbifold χ_y genus, or more generally, the orbifold elliptic genus of an action on a variety are related to the corresponding invariants of arbitrary, not necessarily crepant, resolution of the singularities of the orbifold. This question was addressed in several papers, see for example [4], [7], [15]. The paper [4] contains mathematical definitions of the orbifold E-function as well as an E-function of singular varieties calculated via resolutions, which is called a stringy E-function. The E-function of a smooth manifold is equivalent to the data given by Hodge numbers of the manifold, and it specializes to the χ_y genus. Stringy E-function is defined for singular varieties with log-terminal singularities and more generally for log-terminal pairs. Works [4] and [15] show that the orbifold E-function for a pair (X,G) coincides with the stringy E-function for the pair (X/G, image of ramification divisor). The published version of [4] has a gap in its canonical abelianization algorithm, but it is now corrected by Batyrev [5].

In this paper, two notions of elliptic genus for singular varieties are proposed. The first notion is called *singular elliptic genus* and is defined for pairs (variety, divisor). Singular elliptic genus specializes to the χ_y genus derived from the stringy E-function of [4]. The second notion of elliptic genus, called *orbifold elliptic genus*, is defined for any pair (X,G) of a manifold and a finite group of its automorphisms. Orbifold elliptic genus specializes to the χ_y genus derived from the orbifold E-function. We conjecture that the two elliptic genera coincide for (X/G, image of ramification divisor) and (X,G), up to an explicit normalization factor. The advantage of the orbifold elliptic genus is that it is well-suited for the mathematical proof of the formula (1.1). On the other hand, singular elliptic genus provides an interesting new invariant of singular varieties. As opposed to the non-archimedian integrals over spaces of arcs techniques of [4] and [15], we use the

recent result in factorization of birational maps into a sequence of smooth blowups and blowdowns, see [1].

The content of the paper is as follows. In Section 2, we collect some standard definitions and results that are relevant to the subject but may not be familiar to the reader. In Section 3 we define singular elliptic genus of a $\mathbf{Q}$-Gorenstein complex projective variety Z as follows. If $\pi : Y \to Z$ is a resolution of singularities of Z and $\alpha_k \in \mathbf{Q}$ are defined from the relation $K_Y = \pi^* K_Z + \sum \alpha_k E_k$, then:

$$\widehat{Ell}_Y(Z; z, \tau) := \int_Y \Big(\prod_l \frac{(\frac{y_l}{2\pi \mathrm{i}})\theta(\frac{y_l}{2\pi \mathrm{i}} - z)\theta'(0)}{\theta(-z)\theta(\frac{y_l}{2\pi \mathrm{i}})} \Big) \times \Big(\prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} - (\alpha_k+1)z)\theta(-z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} - z)\theta(-(\alpha_k+1)z)} \Big)$$

where $\theta(z,\tau)$ is the Jacobi theta function, y_l are Chern roots of Y and $e_k = c_1(E_k)$. It is shown that $\widehat{Ell}_Y(Z; z, \tau)$ depends only on Z (rather than on the desingularization Y). Moreover, this definition is extended to pairs (variety, divisor), and singular elliptic genus has transformation properties of a Jacobi form if the pair satisfies a natural Calabi-Yau condition. Some difficulties arise only when some α_k equal (-1), but we generally do not need the log-terminality condition. One application of singular elliptic genus is to the problem raised by M.Goreski and R.McPherson (cf. [9]). They were asking to determine which Chern numbers can be defined for singular spaces so that they are invariant under small resolutions. B.Totaro found a remarkable connection between this problem and the elliptic genus. In [34] he shows that such Chern numbers must be among the combinations of the coefficients of the two variable elliptic genus, by showing that these are the only Chern numbers invariant under the classical flops. As a corollary of our definition of singular elliptic genus, we show that elliptic genera of any two IH-small resolutions (or more generally two crepant resolutions) of a singular variety coincide, which in a sense completes the paper of Totaro. Unfortunately, most varieties do not admit such resolutions, and it appears that Chern numbers may not be a good invariant to look for, because singular elliptic genera generally do not lie in the span of the elliptic genera of smooth varieties. However, coefficients of Taylor expansions of elliptic genera do provide an analog of Chern numbers for singular varieties.

In Section 4 we propose a definition of an orbifold elliptic genus which does not use the resolution of singularities, but uses only information about the manifold and the fixed point sets. Let G be a finite group acting on a manifold X. For $h \in G$, let X^h be a connected component of the fixed point set of h and $TX|_{X^h} = \oplus V_\lambda, \lambda \in \mathbf{Q} \cap [0,1)$ be decomposition into direct sum, such that h acts on V_λ as the multiplication by $e^{2\pi \mathrm{i}\lambda}$. Let $F(h, X^h \subset X) = \sum_\lambda \lambda(h)$ be the fermionic shift (cf. [7], [37]) and:

$$V_{h,X^h \subseteq X} := \otimes_{k\geq 1} \Big[(\Lambda^\bullet V_0^* y q^{k-1}) \otimes (\Lambda^\bullet V_0 y^{-1} q^k) \otimes (Sym^\bullet V_0^* q^k) \otimes (Sym^\bullet V_0 q^k) \otimes$$

$$\otimes \big[\otimes_{\lambda \neq 0} (\Lambda^\bullet V_\lambda^* y q^{k-1+\lambda(h)}) \otimes (\Lambda^\bullet V_\lambda y^{-1} q^{k-\lambda(h)}) \otimes (Sym^\bullet V_\lambda^* q^{k-1+\lambda(h)}) \otimes (Sym^\bullet V_\lambda q^{k-\lambda(h)}) \big] \Big].$$

Then we define (cf. Section 4):

$$Ell_{orb}(X, G; y, q) := y^{-\dim X/2} \sum_{\{h\}, X^h} y^{F(h, X^h \subseteq X)} \frac{1}{|C(h)|} \sum_{g \in C(h)} L(g, V_{h,X^h \subseteq X})$$

where $\{h\}$ is a conjugacy class in G, $C(h)$ is the centralizer of h and $L(g, V_{h,X^h \subseteq X}) = \sum_i (-1)^i \mathrm{tr}(g, H^i(V_{h,X^h \subseteq X}))$ is the holomorphic Lefschetz number. Using the Atiyah-Singer holomorphic Lefschetz theorem, orbifold elliptic genus can be rewritten as

follows. For a pair $g, h \in G$ of commuting elements, let $X^{g,h}$ be a connected component of the set of points in X fixed by both g and h, x_λ be the Chern roots of a subbundle V_λ of $TX|_{X^{g,h}}$ on which both g and h act via the multiplication by $\exp(2\pi \mathrm{i}\lambda(g))$ and $\exp(2\pi \mathrm{i}\lambda(h))$ respectively, and let:

$$\Phi(g, h, \lambda, z, \tau, x) := \frac{\theta(\frac{x}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h))} e^{2\pi \mathrm{i} z \lambda(h) z}.$$

Then:

$$E_{orb}(X, G; z, \tau) = \frac{1}{|G|} \sum_{gh=hg} \Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda \Big) \prod_\lambda \Phi(g, h, \lambda, z, \tau, x_\lambda)[X^{g,h}] \tag{1.5}$$

This formula generalizes (1.2) (as we mentioned already, the latter has as a consequence (1.3), as was shown in [26]). For a thus defined orbifold elliptic genus we prove the formula of Dijkgraaf, Moore, Verlinde and Verlinde (1.1). We also show that if X is a Calabi-Yau manifold, then $E_{orb}(X, G; z, \tau)$ is a weak Jacobi form.

In Section 5 we conjecture (see 5.1) that the two notions of elliptic genera coincide, which would extend the results of [4] and [15]. We prove this conjecture for the toric case and in dimension one. For Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties the elliptic genus was defined already in [12], using the description of the cohomology of chiral de Rham complex $\mathcal{M}SV$ for such hypersurfaces from [10] and borrowing the definition of elliptic genus via chiral de Rham complex in the nonsingular case:

$$Ell(X) = y^{\dim X/2} \mathrm{Supertrace}_{H^*(\mathcal{M}SV(X))} y^{J[0]} q^{L[0]}.$$

Here $\mathcal{M}SV$ is the chiral de Rham complex constructed in [31] and $J[0]$ and $L[0]$ are the operators of the $N = 2$ super-Virasoro algebra acting on $H^*(\mathcal{M}SV(X))$. We use the combinatorial description of this genus, proved in [12], and the calculation of [11] to show that it coincides with the singular elliptic genus, up to an explicit normalization factor.

We continue to discuss Conjecture 5.1 in Section 6. We show that both notions of elliptic genera are invariant under complex cobordisms of G action. By using the known result about cobordism classes of the action of a cyclic group of prime order p, we prove Conjecture 5.1 for involutions.

The authors wish to thank Burt Totaro for his helpful comments.

2. Preliminaries

2.1. **Elliptic genus.** Let X be a compact (almost complex) manifold. For a bundle V on X we consider the following elements in the ring of formal power series over $K(X)$:

$$S_t(V) = \sum_i S^i(V) t^i \quad \Lambda_t(V) = \sum_i \Lambda^i(V) t^i$$

where S^i (resp. Λ^i) is the i-th symmetric (resp. exterior) power of V.
Let T_X (resp. $\bar{T}_X$) be tangent (resp. cotangent) bundle. The elliptic genus of X can be defined as:

$$Ell(X; y, q) = \int_X ch(\mathcal{E}LL_{y,q}) td(X)$$

where

$$\mathcal{E}LL_{y,q} := y^{-\frac{d}{2}} \otimes_{n \geq 1} \Big(\Lambda_{-yq^{n-1}} \bar{T}_X \otimes \Lambda_{-y^{-1}q^n} T_X \otimes S_{q^n} \bar{T}_X \otimes S_{q^n} T_X \Big).$$

If x_i are the Chern roots of X, i.e. for the total Chern class we have $c(X) = \prod_i (1+x_i)$, then

$$Ell(X;y,q) = \int_X \prod_i x_i \frac{\theta(\frac{x_i}{2\pi \mathrm{i}} - z, \tau)}{\theta(\frac{x_i}{2\pi \mathrm{i}}, \tau)} \tag{2.1}$$

where $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ and $y = \mathrm{e}^{2\pi \mathrm{i} z}$. In (2.1)

$$\theta(z,\tau) = q^{\frac{1}{8}}(2\sin \pi z) \prod_{l=1}^{l=\infty}(1-q^l) \prod_{l=1}^{l=\infty}(1-q^l \mathrm{e}^{2\pi \mathrm{i} z})(1-q^l \mathrm{e}^{-2\pi \mathrm{i} z})$$

is the Jacobi theta-function ([13]).

For $q=0$ we have: $Ell(X;y,q=0) = y^{-\frac{d}{2}}\chi_{-y}(X)$ where

$$\chi_y(X) = \sum_{p,q} (-1)^q \dim H^q(X, \Omega_X^p) y^p$$

is Hirzebruch χ_y-genus (cf. [25]). In particular, $Ell(X; y=1, q=0)$ is the topological Euler characteristic, $(-1)^{d/2} Ell(X; y=-1, q=0)$ is the signature, etc.

If X is a Calabi Yau, i.e. $K_X \sim 0$, then $Ell(X;y,q)$ is a weak Jacobi form. Recall (cf. [18], [23]) that weight $k \in \mathbf{Z}$ and index $r \in \frac{1}{2}\mathbf{Z}$ weak Jacobi form is a function on $H \times \mathbf{C}$ that satisfies:

$$\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{2\pi i \frac{rcz^2}{c\tau+d}} \phi(\tau,z)$$

$$\phi(\tau, z+m\tau+n) = (-1)^{2r(\lambda+\mu)} e^{-2\pi i r(m^2\tau+2mz)} \phi(\tau,z)$$

and has a Fourier expansion $\sum_{l,m} c_{m,l} y^l q^m$ with nonnegative m.

2.2. **Log-terminal singularities.** We recall basic definitions related to singular varieties. Let Z be a normal irreducible projective variety. $\mathbf{Q}$-Weil (resp. $\mathbf{Q}$-Cartier) divisor is a linear combination with rational coefficients of codimension one subvarieties (resp. Cartier divisors) on Z.

The canonical divisor K_Z of Z is a Weil divisor $div(s)$ where $s = df_1 \wedge \ldots \wedge df_{dim Z}$ (f_i are meromorphic functions) is a non zero rational differential on Z. We call Z Gorenstein (resp. $\mathbf{Q}$-Gorenstein) is K_Z is Cartier (resp. $\mathbf{Q}$-Cartier).

A resolution of singularities of a variety Z is a proper birational morphism $f: Y \to Z$ where Y is smooth.

Definition 2.1. (cf. [9]) An IH-small resolution of Z is a regular map $Y \to Z$ such that for every $i \geq 1$ the set of points $z \in Z$ such that $\dim(f^{-1}(z)) \geq i$ has codimension greater than $2i$ in Z.

Definition 2.2. *Z has at worst log-terminal singularities if the following two conditions hold.*
(i) Z is $\mathbf{Q}$-Gorenstein.
(ii) For a resolution $f: X \to Z$, whose exceptional set is a divisor with simple normal crossings, in the relation $K_X = f^*K_Z + \sum \alpha_i E_i$ one has $\alpha_i > -1$.

A well-known result of birational geometry, see for example [28], states that for any resolution of a log-terminal variety Z, the coefficients α_i (called *discrepancies*) are bigger than (-1). Similar definition of log-terminality exists for pairs (Z,D) where D is a $\mathbf{Q}$-Weil divisor on a normal variety Z such that (K_Z+D) is $\mathbf{Q}$-Cartier.

2.3. **G-bundles.** Let X be a complex manifold and G a finite group of holomorphic transformations acting on X. Let V be a holomorphic G-bundle on X, i.e. the action of G on X is extended to the action on V. The holomorphic Lefschetz number of $g \in G$ is

$$L(g,V) = \sum_i (-1)^i \mathrm{tr}(g, H^i(X,V))$$

Let V^G be the sheaf which sections over open sets are the G-invariants of the sections of V. We have (cf. [24], spectral sequences degenerate due to finiteness of G):

$$\chi(V^G) = \frac{1}{|G|}\sum_{g\in G} L(g,V).$$

The Lefschetz numbers are given by the data around the fixed point sets (cf. [2]) as follows. Let N^g be the normal bundle to the fixed point set X^g of g, and let N^{g*} be its dual. In the case when the action of G on a space Y is trivial, we have $K_G(Y) = K(Y) \otimes R(G)$ (cf. [2]) and hence one can define $W(g) \in K(Y)$ corresponding to $W \in K_G(Y)$. In these notations:

$$L(g,V) = \frac{chV|_{X^g}(g)td(T_{X^g})}{ch\Lambda_{-1}(N^g)^*(g)}[X^g]. \tag{2.2}$$

For $g \in G$ the normal bundle N_{X^g} to the fixed point set X^g can be decomposed into direct sum $N_{X^g} = \oplus_i N(\theta_i)$, $\theta_i \in \mathbf{Q}$ where each $N(\theta_i)$ is the subbundle on which g acts as multiplication by $e^{2\pi i\theta_i}$. If $x_{\theta_i,j}$ are the Chern roots of $N(\theta_i)$ i.e. $c(N(\theta_i)) = \prod_j (1 + x_{\theta_i,j})$ then (2.2) can be rewritten as:

$$L(g,V) = \frac{ch(V|_{X^g})}{\prod_{i,j}(1 - e^{-x_j - \theta_{i,j}})} td(X^g)[X^g].$$

3. Singular elliptic genus

In this section we define *singular elliptic genus* for a large class of singular varieties and more generally for pairs consisting of a variety and a $\mathbf{Q}$-Cartier divisor on it. This is by far the most general definition of elliptic genus for singular varieties constructed to date. All varieties are assumed to be proper over $\mathrm{Spec}(\mathbf{C})$.

Definition 3.1. Let Z be a $\mathbf{Q}$-Gorenstein variety with log-terminal singularities, and let $\pi : Y \to Z$ be a desingularization of Z whose exceptional divisor $E = \sum_k E_k$ has simple normal crosings. The discrepancies α_k of the components E_k are determined by the formula

$$K_Y = \pi^* K_Z + \sum_k \alpha_k E_k.$$

We introduce Chern roots y_l of Y by $c(TY) = \prod_l (1 + y_l)$ and define cohomology classes $e_k := c_1(E_k)$. *Singular elliptic genus* of Z is then defined as a function of two variables z, τ given by

$$\widehat{Ell}_Y(Z; z,\tau) := \int_Y \Big(\prod_l \frac{(\frac{y_l}{2\pi \mathrm{i}})\theta(\frac{y_l}{2\pi \mathrm{i}} - z)\theta'(0)}{\theta(-z)\theta(\frac{y_l}{2\pi \mathrm{i}})}\Big) \times \Big(\prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} - (\alpha_k+1)z)\theta(-z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} - z)\theta(-(\alpha_k+1)z)}\Big)$$

where $\theta(z,\tau)$ is the Jacobi theta function, see [13]. We will often suppress the τ-dependence in our formulas.

We will usually abuse notation and consider $\widehat{Ell}$ to be a function of $y = \mathrm{e}^{2\pi\mathrm{i}z}$ and $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. Strictly speaking, this function will be multi-valued, because rational powers of y may occur.

The key result of this section is the following theorem.

Theorem 3.2. *The above defined $\widehat{Ell}_Y(Z; y, q)$ does not depend on the choice of desingularization Y and therefore defines an invariant of Z, which we simply denote by $\widehat{Ell}(Z; y, q)$.*

Proof. Because of the Weak Factorization Theorem of [1] it suffices to show that $\widehat{Ell}_Y(Z; y, q) = \widehat{Ell}_{\tilde{Y}}(Z; y, q)$ when $\tilde{Y}$ is obtained from Y by a blowup along a nonsingular subvariety X. We remark that the algorithm of [1] is compatible with the normal crossing condition (cf. [1], Theorem 0.3.1), so we may assume that X has normal crossings with the components of the exceptional divisor of $\pi : Y \to Z$.

We will use the notations of Fulton [21] for the blowup diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{Y} \\ g\downarrow & & \downarrow f \\ X & \xrightarrow[i]{} & Y \end{array}$$

where $\tilde{X}$ is the exceptional divisor of the blowdown morphism. We also have $\pi : Y \to Z$ and $\pi \circ f : \tilde{Y} \to Z$. Discrepancies of the exceptional divisors of these morphisms are related by

$$K_Y = \pi^* K_Z + \sum_k \alpha_k E_k$$

$$K_{\tilde{Y}} = f^*\pi^* K_Z + \sum_k \alpha_k E'_k + (\sum_k \alpha_k \beta_k + r - 1)\tilde{X}$$

where β_k is the multiplicity of E_k along X and r is the codimension of X in Y.

We will use for a while the following technical assumption.

(3.1) The normal bundle N to X inside Y is a pullback under i of some rank r bundle M on Y.

We have the following exact sequences of coherent sheaves on $\tilde{Y}$, see Section 15.4 of [21].

$$0 \to T\tilde{Y} \to f^*TY \to j_*F \to 0$$
$$0 \to j_*\mathcal{O}_{\tilde{X}}(-1) \to j_*g^*i^*M \to j_*F \to 0$$
$$0 \to \mathcal{O} \to \mathcal{O}(\tilde{X}) \to j_*\mathcal{O}_{\tilde{X}}(-1) \to 0$$
$$0 \to f^*M(-\tilde{X}) \to f^*(M) \to j_*g^*i^*M \to 0$$

Here F is the tautological quotient bundle on $\tilde{X}$. This implies

$$c(T\tilde{Y}) = c(f^*TY) \cdot (1 + \tilde{x}) \cdot \prod_l \frac{(1 + f^*m_l - \tilde{x})}{(1 + f^*m_l)}$$

where $c(M) = \prod_l (1 + m_i)$ and $\tilde{x} = c_1(\mathcal{O}(\tilde{X}))$. Note also that $c_1(E'_k) = f^*e_k - \beta_k\tilde{x}$. Therefore,

$$\widehat{Ell}_{\tilde{Y}}(Z; y, q) = \int_{\tilde{Y}} \Big(\prod_l \frac{(\frac{f^*y_l}{2\pi\mathrm{i}})\theta(\frac{f^*y_l}{2\pi\mathrm{i}} - z)\theta'(0)}{\theta(-z)\theta(\frac{f^*y_l}{2\pi\mathrm{i}})}\Big) \times \Big(\frac{(\frac{\tilde{x}}{2\pi\mathrm{i}})\theta(\frac{\tilde{x}}{2\pi\mathrm{i}} - z)\theta'(0)}{\theta(-z)\theta(\frac{\tilde{x}}{2\pi\mathrm{i}})}\Big) \times$$

$$\times\Big(\prod_l \frac{\theta(\frac{f^*m_l-\tilde{x}}{2\pi\mathrm{i}}-z)(\frac{f^*m_l-\tilde{x}}{2\pi\mathrm{i}})\theta(\frac{f^*m_l}{2\pi\mathrm{i}})}{\theta(\frac{f^*m_l-\tilde{x}}{2\pi\mathrm{i}})(\frac{f^*m_l}{2\pi\mathrm{i}})\theta(\frac{f^*m_l}{2\pi\mathrm{i}}-z)}\Big)\times\Big(\prod_k \frac{\theta(\frac{f^*e_k-\beta_k\tilde{x}}{2\pi\mathrm{i}}-(\alpha_k+1)z)\theta(-z)}{\theta(\frac{f^*e_k-\beta_k\tilde{x}}{2\pi\mathrm{i}}-z)\theta(-(\alpha_k+1)z)}\Big)\times$$
$$\times\Big(\frac{\theta(\frac{\tilde{x}}{2\pi\mathrm{i}}-(\alpha_{\tilde{X}}+1)z)\theta(-z)}{\theta(\frac{\tilde{x}}{2\pi\mathrm{i}}-z)\theta(-(\alpha_{\tilde{X}}+1)z)}\Big)$$

where $\alpha_{\tilde{x}}=r-1+\sum_k\alpha_k\beta_k$.

We will now use $\int_{\tilde{Y}}a=\int_Y f_*(a)$. We write the Taylor expansion $\sum_n R_n(y,q)\tilde{x}^n$ of the expression under $\int_{\tilde{Y}}$ in the above identity. Observe that $f_*R_0(y,q)$ is exactly the class in $A(Y)$ whose integral is $\widehat{Ell}_Y(Z;y,q)$, so we need to show that the contribution of the rest of the terms vanishes. Notice that $f_*\tilde{x}^n=0$ for $1\le n\le r-1$ and $f_*\tilde{x}^{r+n}=i_*(s_k(i^*M))(-1)^{n+r-1}$ where $\sum_{n\ge 0}s_nt^n$ is Segre polynomial of a vector bundle, see [21]. Hence, one needs to calculate

$$\int_Y\sum_{n\ge 0}i_*s_n(i^*M)(-1)^{n+r-1}\cdot(\text{Coeff. at }t^{r+n})\Big[\Big(\prod_l\frac{(\frac{y_l}{2\pi\mathrm{i}})\theta(\frac{y_l}{2\pi\mathrm{i}}-z)\theta'(0)}{\theta(-z)\theta(\frac{y_l}{2\pi\mathrm{i}})}\Big)\times$$
$$(3.2)\qquad \times\Big(\frac{(\frac{t}{2\pi\mathrm{i}})\theta(\frac{t}{2\pi\mathrm{i}}-(\alpha_{\tilde{X}}+1)z)\theta'(0)}{\theta(\frac{t}{2\pi\mathrm{i}})\theta(-(\alpha_{\tilde{X}}+1)z)}\Big)\times\Big(\prod_l\frac{\theta(\frac{m_l-t}{2\pi\mathrm{i}}-z)(\frac{m_l-t}{2\pi\mathrm{i}})\theta(\frac{m_l}{2\pi\mathrm{i}})}{\theta(\frac{m_l-t}{2\pi\mathrm{i}})(\frac{m_l}{2\pi\mathrm{i}})\theta(\frac{m_l}{2\pi\mathrm{i}}-z)}\Big)\times$$
$$\times\Big(\prod_k\frac{\theta(\frac{e_k-\beta_kt}{2\pi\mathrm{i}}-(\alpha_k+1)z)\theta(-z)}{\theta(\frac{e_k-\beta_kt}{2\pi\mathrm{i}}-z)\theta(-(\alpha_k+1)z)}\Big)\Big].$$

We denote $n_l=i^*m_l$, $f_k=i^*e_k$ and use the fact that

$$\sum_{n\ge 0}s_n(i^*M)(-1)^nt^{-k}=\frac{t^r}{\prod_l(t-n_l)}$$

to rewrite (3.2) as

$$\text{const.}\int_X(\text{Coeff. at }t^{-1})\Big[\Big(\prod_l\frac{(\frac{x_l}{2\pi\mathrm{i}})\theta(\frac{x_l}{2\pi\mathrm{i}}-z)\theta'(0)}{\theta(-z)\theta(\frac{x_l}{2\pi\mathrm{i}})}\Big)\times$$
$$(3.3)\qquad \times\Big(\frac{\theta(\frac{t}{2\pi\mathrm{i}}-(\alpha_{\tilde{X}}+1)z)\theta'(0)}{\theta(\frac{t}{2\pi\mathrm{i}})\theta(-(\alpha_{\tilde{X}}+1)z)}\Big)\times\Big(\prod_l\frac{\theta(\frac{n_l-t}{2\pi\mathrm{i}}-z)\theta(\frac{n_l}{2\pi\mathrm{i}})}{\theta(\frac{n_l-t}{2\pi\mathrm{i}})(\frac{n_l}{2\pi\mathrm{i}})\theta(\frac{n_l}{2\pi\mathrm{i}}-z)}\Big)\times$$
$$\times\Big(\prod_k\frac{\theta(\frac{f_k-\beta_kt}{2\pi\mathrm{i}}-(\alpha_k+1)z)\theta(-z)}{\theta(\frac{f_k-\beta_kt}{2\pi\mathrm{i}}-z)\theta(-(\alpha_k+1)z)}\Big)\Big].$$

Here we denote $c(TX)=\prod_l(1+x_l)$ and use $c(TX)=i^*c(TY)/i^*c(M)$. To show that (3.3) is zero, observe that the function whose coefficient at t^{-1} is measured, is elliptic in t. Really, $t\to t+2\pi\mathrm{i}$ obviously keeps it unchanged, and $t\to t+2\pi\mathrm{i}\tau$ does not change it, because of $\alpha_{\tilde{X}}=\sum_k\alpha_k\beta_k+r-1$. We have used here the fact that none of the α-s is equal to (-1), which follows from the condition that Z is log-terminal, see for instance [28]. It remains to show that $t=0$ is the only pole of this function up to the lattice $2\pi\mathrm{i}(\mathbf{Z}+\mathbf{Z}\tau)$, so the residue is zero. To do so, observe that the normal crossing condition implies $\beta_k\in\{0,1\}$, and moreover, whenever $\beta_k=1$ the corresponding factor $\theta(\frac{f_k-t}{2\pi\mathrm{i}}-z)$ in the denominator of the last product is offset by a factor $\theta(\frac{n_l-t}{2\pi\mathrm{i}}-z)$ in the numerator of the second product.

We will now get rid of the assumption (3.1). Indeed, it is easy to see that the difference between $\widehat{Ell}_Y(Z;y,q)$ and $\widehat{Ell}_{\tilde{Y}}(Z;y,q)$ can be written as a degree of an

element of $A(\tilde{X})$ that is preserved when one deforms $i : X \to Y$ to the embedding of X into the normal cone, for which the assumption (3.1) is satisfied. $\square$

Remark 3.3. We have not significantly used the log-terminality condition, except for the fact that we did not have to divide by $\theta(0 \cdot z)$. Therefore, singular elliptic genera can in fact be defined for all varieties that admit a resolution with no (-1) discrepancies. In fact, we will extend our definition to the category of pairs that consist of an algebraic variety and a $\mathbf{Q}$-Cartier divisor on it.

Definition 3.4. Let Z be a projective variety, and let D be an arbitrary $\mathbf{Q}$-Weil divisor such that $K_Z + D$ is a $\mathbf{Q}$-Cartier divisor on Z. Let $\pi : Y \to Z$ be a desingularization of Z. We denote by $E = \sum_k E_k$ the exceptional divisor of π plus the sum of the proper preimages of the components of D and assume that it has simple normal crossings. The discrepancies α_k of the components E_k are determined by the formula

$$K_Y = \pi^*(K_Z + D) + \sum_k \alpha_k E_k$$

and the requirement that the discrepancy of the proper transform of a component of D is the opposite of the coefficient of D at that component. We introduce Chern roots y_l of Y by $c(TY) = \prod_l (1 + y_l)$ and define

$$\widehat{Ell}_Y(Z, D; y, q) := \int_Y \Big(\prod_l \frac{(\frac{y_l}{2\pi \mathrm{i}})\theta(\frac{y_l}{2\pi \mathrm{i}} - z)\theta'(0)}{\theta(-z)\theta(\frac{y_l}{2\pi \mathrm{i}})}\Big) \times \Big(\prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} - (\alpha_k + 1)z)\theta(-z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} - z)\theta(-(\alpha_k + 1)z)}\Big)$$

where as usual $y = \mathrm{e}^{2\pi \mathrm{i} z}$, $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, the τ-dependence is suppressed, and $e_k = c_1(\mathcal{O}(E_k))$. If some of the discrepancies α_k equal (-1), then we try to define the elliptic genus as follows. For any ample effective Cartier divisor H on Z that contains all singular points of Z and all components of D we calculate

$$\lim_{n \to \infty} \widehat{Ell}_Y(Z, D + H/n; z, \tau)$$

for each (z, τ). If such limit exists and is independent of H, then we call it $\widehat{Ell}_Y(Z, D; z, \tau)$. Notice that if n is sufficiently big, then the discrepancies of all divisors E_k, calculated for the pair $(Z, D + H/n)$ are not equal to (-1). It is also easy to see that if there are no (-1) discrepancies, then $\lim_{n\to\infty} \widehat{Ell}_Y(Z, D + H/n; z, \tau) = \widehat{Ell}_Y(Z, D; z, \tau)$.

Theorem 3.5. *The above defined elliptic genus does not depend on the choice of the desingularization $\pi : Y \to Z$. We will therefore denote it simply by $\widehat{Ell}(Z, D; y, q)$.*

Proof. Any two resolutions of singularities of Z can be connected by a sequence of blowups and blowdowns. Let Y and $\tilde{Y}$ be two resolutions of Z, such that $\tilde{Y}$ is the blowup of Y as in the proof of Theorem 3.2. For any H and n big enough to assure that all discrepancies are not equal to (-1), the proof of Theorem 3.2 implies that

$$\widehat{Ell}_Y(Z, D + H/n; z, \tau) = \widehat{Ell}_{\tilde{Y}}(Z, D + H/n; z, \tau).$$

Then elliptic genera defined via Y and $\tilde{Y}$ coincide by definition. $\square$

Remark 3.6. The reason behind extending the definition of the elliptic genus via above limits is the following. In the non-log-terminal case, it is conceivable that there exist two resolutions of singularities without (-1) discrepancies that can only be connected via resolutions with (-1) discrepancies. The above theorem assures that elliptic genera defined via such resolutions are the same.

Proposition 3.7. *Elliptic genera of two different crepant resolutions of a Gorenstein projective variety coincide.*

Proof. We will show that elliptic genus of a crepant resolution Y of a variety X equals to the singular elliptic genus of X. If the exceptional set of the morphism $\pi : Y \to X$ is a divisor with simple normal crossings, then it is enough to observe that in the Definition 3.1 the second product is trivial. In general, we can further blow up Y to get $\mu : Z \to Y$ so that the exceptional sets of μ and $\pi \circ \mu : Z \to X$ are divisors with simple normal crossings. Then singular elliptic genera of Y and X calculated via Z are given by the same formula, because the discrepancies coincide. $\square$

Remark 3.8. In particular, the above proposition shows that the statement of Theorem 8.1 of [34] can be extended to the full elliptic genus.

The following proposition shows that when $q \to 0$, we recover a formula for χ_y genus of (Z, D) which follows from [4].

Proposition 3.9. *Let (Z, D) be a log-terminal pair, see* [4]*. Then*

$$\widehat{Ell}(Z, D; u, q = 0) = (u^{-\frac{1}{2}} - u^{\frac{1}{2}})^{\dim Z} E_{st}(Z, D; u, 1)$$

where E_{st} is defined in [4].

Proof. To avoid confusion, we immediately remark that the second arguments in singular elliptic genus and in Batyrev's E-function have drastically different meanings. The definition of $E_{st}(Z, D)$ in [4] could be stated as

$$E_{st}(Z, D; u, v) := \sum_{J \subset I} E(E_J; u, v) \prod_{j \in J} \Big(\frac{uv - 1}{uv^{\alpha_j + 1} - 1} - 1 \Big)$$

where $\sum_{i \in I} \alpha_i E_i$ is the exceptional divisor of a resolution $Y \to Z$ together with proper preimages of the components of D, and is assumed to have normal crossings. Polynomials $E(E_J; u, v)$ are defined in terms of mixed Hodge structure on the cohomology of E_J, see [4]. Subvariety E_J is $\cap_{j \in J} E_j$, and the sum includes the empty subset J.

For each J

$$E_{st}(E_J; u, 1) = \int_{E_j} \prod_{i=1}^{\dim E_J} \frac{(1 - u\mathrm{e}^{-x_{i,J}}) x_{i,J}}{1 - \mathrm{e}^{-x_{i,J}}},$$

where $c(TE_J) = \prod_i (1 + x_i, J)$. The adjunction formula for complete intersections yields

$$c(TE_J) = i_J^*(c(TY)) / \prod_{j \in J} (1 + i_J^* c_1(E_j)),$$

where $i_J : E_J \to Y$ is the closed embedding. We then obtain

$$E(E_J; u, 1) = \int_{E_j} \prod_{i=1}^{\dim Y} \frac{(1 - u\mathrm{e}^{-i_J^* x_i}) i_J^* x_i}{1 - \mathrm{e}^{-i_J^* x_i}} \prod_{j \in J} \frac{1 - \mathrm{e}^{-i_J^* e_j}}{(1 - u\mathrm{e}^{-i_J^* e_j}) i_J^* e_j}$$

$$= \int_Y \prod_{i=1}^{\dim Y} \frac{(1-u\mathrm{e}^{-x_i})x_i}{1-\mathrm{e}^{-x_i}} \prod_{j\in J} \frac{1-\mathrm{e}^{-e_j}}{1-u\mathrm{e}^{-e_j}}$$

where $c(TY) = \prod_i (1+x_i)$. When we plug this result into Batyrev's formula, we get

$$\begin{aligned} E_{st}(Z,D;u,1) &= \int_Y \prod_{i=1}^{\dim Y} \frac{(1-u\mathrm{e}^{-x_i})x_i}{1-\mathrm{e}^{-x_i}} \prod_{j\in I}\Big(1+\frac{(u-u^{\alpha_j+1})(1-\mathrm{e}^{-e_j})}{(u^{\alpha_j+1}-1)(1-u\mathrm{e}^{-e_j})}\Big) \\ &= \int_Y \prod_{i=1}^{\dim Y} \frac{(1-u\mathrm{e}^{-x_i})x_i}{1-\mathrm{e}^{-x_i}} \prod_{j\in I} \frac{(u-1)(1-u^{\alpha_j+1}\mathrm{e}^{-e_j})}{(u^{\alpha_j+1}-1)(1-u\mathrm{e}^{-e_j})} \\ &= (u^{-\frac{1}{2}}-u^{\frac{1}{2}})^{-\dim Z} \lim_{q\to 0} \widehat{Ell}(Z,D;u,q). \end{aligned}$$

□

The following simple proposition establishes modular properties of singular elliptic genus in Calabi-Yau case.

Proposition 3.10. *Let (Z,D) be a Calabi-Yau pair, in the sense that $K_Z + D$ is zero as a $\mathbf{Q}$-Cartier divisor. If Z admits a resolution such that all the discrepancies of (Z,D) are not equal to (-1), then singular elliptic genus $\widehat{Ell}(Z,D;y,q)$ has transformation properties of Jacobi form of weight $\dim Z$ and index 0 for the subgroup of the full Jacobi group generated by*

$$(z,\tau)\to(z+n,\tau),\ (z,\tau)\to(z+n\tau,\tau),\ (z,\tau)\to(z,\tau+1),\ (z,\tau)\to(z/\tau,-1/\tau)$$

where n is the least common denominator of the discrepancies.

Proof. Transformation properties of $\theta(z,\tau)$ under $(z,\tau)\to(z+1,\tau)$ and $(z,\tau)\to(z+\tau,\tau)$ together with Calabi-Yau condition

$$K_Y = \sum_k \alpha_k E_k$$

assure that

$$\widehat{Ell}(Z,D;z+n,\tau) = \widehat{Ell}(Z,D;z+n\tau,\tau) = \widehat{Ell}(Z,D;z,\tau).$$

We needed here that $n\alpha_k \in \mathbf{Z}$. Similarly, the transformation properties of θ under $(z,\tau)\to(z,\tau+1)$ show that

$$\widehat{Ell}(Z,D;z,\tau+1) = \widehat{Ell}(Z,D;z,\tau).$$

It remains to investigate what happens under $(z,\tau)\to(z/\tau,-1/\tau)$. For this, one considers the change $(e_k,y_l)\to(e_k/\tau,y_l/\tau)$ in the formula of the Definition 3.4. A rather lengthy but straightforward calculation, similar to that of Theorem 2.2 of [12], shows that

$$\widehat{Ell}(Z,D;\frac{z}{\tau},-\frac{1}{\tau}) = \tau^{\dim Z}\widehat{Ell}(Z,D;z,\tau).$$

□

Another application of our techniques is the following theorem, which complements similar results for Hodge numbers of Calabi-Yau manifolds, see for example [6] and [16].

Theorem 3.11. *Elliptic genera of two birationally equivalent Calabi-Yau manifolds coincide. Moreover, the statement is true for smooth projective algebraic manifolds X with $nK_X \sim 0$ for some n.*

Proof. Let Z_1 and Z_2 be two birationally equivalent Calabi-Yau manifolds or their generalizations above. Let Y be a desingularization of the closure of the graph of the birational equivalence, so that $\pi_{1,2} : Y \to Z_{1,2}$ are regular birational morphisms. Let n be the smallest integer so that $nK_{Z_{1,2}}$ is rationally equivalent to zero, and therefore has a global section. Global sections of the pluricanonical bundle are birational invariants, so one can consider the divisor $\sum_k a_k E_k$ of this section on Y. It is easy to see that for both morphisms π_1 and π_2 the exceptional divisor is $\sum_k (a_k/n) E_k$, which we can then assume to have simple normal crossings (perhaps by passing to a new desingularization). Therefore, elliptic genera of $Z_{1,2}$ are calculated on Y using the same discrepancies. □

Remark 3.12. It is interesting to compare the results of this section with the work of Totaro in [34], who tried to see which Chern numbers can be meaningfully defined for singular varieties. For varieties that admit IH-small resolutions, singular elliptic genus provides the maximum possible collection of such numbers. Totaro has shown that every flop-invariant Chern number comes from the elliptic genus and obtained partial results in the opposite direction by means of intersection cohomology.

In general, coefficients of the singular elliptic genus of Z at $y^k q^l$ provide analogs of Chern numbers of singular varieties in the following sense.
1. They are the invariants of isomorphism class of singular spaces.
2. For manifolds these invariants are the usual Chern numbers (i.e. linear combinations of $c_{i_1}(X) \cdots c_{i_N}(X)[X]$ where $\sum_{k=1}^{k=N} i_k = \dim X$ and $[X]$ is the fundamental class of a manifold X).
3. These invariants are unchanged under small resolutions.

In fact, for singular varieties, elliptic genera may contain more information than in the non-singular case. For varieties with non-Gorenstein singularities, singular elliptic genus may depend on rational powers of y. Moreover, there exist examples of Gorenstein varieties whose elliptic genera do not lie in the span of elliptic genera of nonsingular varieties. This can be seen already at the level of χ_y genus, see an example in [3] of a variety with Gorenstein canonical singularities whose E-function is not a polynomial.

We hope that elliptic genera of singular varieties can be interpreted as nontrivial invariants of not yet defined cobordism theory of singular spaces. Transformations leaving the singular elliptic genus invariant in such theory for smooth manifolds should include usual cobordisms as well as flops. It would be interesting to compare our results with the invariants of Witt spaces studied by P.Siegel, the latter however were defined in SO rather than in complex category (cf. [9], [33]).

Remark 3.13. We do not have a good understanding of the reason why (-1) discrepancies seem to be a problem. One can observe, however, that in the case of a surface singularity obtained by contracting a single smooth curve on a smooth surface to a point, the discrepancy is (-1) if and only if the curve in question is elliptic.

4. Orbifold elliptic genus and DMVV formula

In this section we define *orbifold elliptic genus*, which we conjecture to equal the singular elliptic genus of Section 3. We delay the comparison of two genera until Section 5. Instead, the goal of this section is to show how this definition of orbifold elliptic genus allows one to recover the formula of [17] whose derivation was based partly on heuristic string-theoretic arguments. Our definition of elliptic genus is inspired by the calculations of [10].

Definition 4.1. Let X be an algebraic variety acted upon by a finite group G. We assume that the subgroup of elements of G acting trivially on X contains only the identity. We define the following function of two variables that we call *orbifold elliptic genus* of X/G:

$$Ell_{orb}(X,G;y,q) := y^{-\dim X/2} \sum_{\{h\},X^h} y^{F(h,X^h\subseteq X)} \frac{1}{|C(h)|} \sum_{g\in C(h)} L(g, V_{h,X^h\subseteq X})$$

where $F(h,X^h\subseteq X)$ is the fermionic shift (cf. [37], [7]) and $V_{h,X^h\subseteq X}$ is a vector bundle over X^h defined as follows. Let $TX|_{X^h}$ decompose into eigensheaves for h as

$$V_0 \oplus (\oplus_{\lambda:<h>\to\mathbf{Q}/\mathbf{Z}} V_\lambda). \tag{4.1}$$

We lift $\lambda(h)$ to a rational number in $[0,1)$. Then $V_{h,X^h\subseteq X}$ is defined as

$$V_{h,X^h\subseteq X} := \otimes_{k\geq 1}\Big[(\Lambda^\bullet V_0^* y q^{k-1})\otimes(\Lambda^\bullet V_0 y^{-1}q^k)\otimes(Sym^\bullet V_0^* q^k)\otimes(Sym^\bullet V_0 q^k)\otimes$$
$$\otimes\big[\otimes_{\lambda\neq 0}(\Lambda^\bullet V_\lambda^* y q^{k-1+\lambda(h)})\otimes(\Lambda^\bullet V_\lambda y^{-1} q^{k-\lambda(h)})\otimes(Sym^\bullet V_\lambda^* q^{k-1+\lambda(h)})\otimes(Sym^\bullet V_\lambda q^{k-\lambda(h)})\big]\Big].$$

Remark 4.2. Another way to state this definition is

$$Ell_{orb}(X,G;y,q) := y^{-\dim X/2} \sum_{\{h\},X^h} y^{F(h,X^h\subseteq X)} \chi(H^\bullet(V_{h,X^h\subseteq X}^{C(h)})).$$

Theorem 4.3. *Let X and G be as above and let $X^{g,h}$ be the set of fixed points of a pair of commuting elements $g,h\in G$. Let $TX|_{X^{g,h}} = \oplus W_\lambda$ be the decomposition (refinement of (4.1)) of the restriction on $X^{g,h}$ of the tangent bundle into direct sum of line bundles on which g (resp. h) acts as multiplication by $\mathrm{e}^{2\pi \mathrm{i}\lambda(g)}$ (resp. $\mathrm{e}^{2\pi\mathrm{i}\lambda(h)}$). Denote by x_λ the Chern roots of the bundle W_λ.*

1. We have:

$$Ell_{orb}(X,G) = \frac{1}{|G|}\sum_{g,h,gh=hg}\Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda\Big)\prod_\lambda \frac{\theta(\tau,\frac{x_\lambda}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h)-z)}{\theta(\tau,\frac{x_\lambda}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h))}\mathrm{e}^{2\pi\mathrm{i}\lambda(h)z}[X^{g,h}].$$

2. Let X be a Calabi-Yau of dimension d, such that $H^0(X,K_X)=\mathbf{C}$. Denote by n the order of G in $\mathrm{Aut}H^0(X,K_X)$. Then $Ell_{orb}(X,G)$ is a weak Jacobi form of weight 0 and index $d/2$ with respect to subgroup of the Jacobi group Γ^J generated by transformations

$$(z,\tau)\to(z+n,\tau),\ (z,\tau)\to(z+n\tau,\tau),\ (z,\tau)\to(z,\tau+1),\ (z,\tau)\to(\frac{z}{\tau},-\frac{1}{\tau}).$$

In particular, if the action preserves holomorphic volume then $Ell_{orb}(X,G)$ is a weak Jacobi form of weight 0 and index $d/2$ for the full Jacobi group.

Proof. We replace the contribution of each conjugacy class by an average contribution of its elements to obtain

$$Ell_{orb}(X,G)=\frac{1}{|G|}y^{-\dim X/2}\sum_{gh=hg}y^{F(h,X^h\subset X)}L(g,V_{h,X^h\subset X}).$$

Using holomorphic Lefschetz theorem, we obtain:

$$Ell_{orb}(X,G)=\frac{1}{|G|}y^{-\dim X/2}\sum_{gh=hg}y^{F(h,X^h\subset X)}\frac{ch(V_{h,X^h\subset X}|_{X^{g,h}})(g)td(T_{X^{g,h}})[X^{g,h}]}{ch\Lambda_{-1}(N^g_{X^h})^*(g)},$$

where $N^g_{X^h}$ is a the normal bundle to $X^{g,h}$ in X^h. An explicit calculation of the Chern and Todd classes then yields

$$Ell_{orb}(X,G)=\frac{1}{|G|}\sum_{gh=hg}y^{F(h,X^h\subset X)-\dim X/2}\Big(\prod_{\lambda(g)=\lambda(h)=0}x_\lambda\Big)\times$$

$$\times\prod_{k\geq 1,\lambda}\frac{(1-yq^{k-1+\lambda(h)}e^{-x_\lambda-2\pi \mathrm{i}\lambda(g)})(1-y^{-1}q^{k-\lambda(h)}e^{x_\lambda+2\pi\mathrm{i}\lambda(g)})}{(1-q^{k-1+\lambda(h)}e^{-x_\lambda-2\pi\mathrm{i}\lambda(g)})(1-q^{k-\lambda(h)}e^{x_\lambda+2\pi\mathrm{i}\lambda(g)})}=$$

$$\frac{1}{|G|}\sum_{gh=hg}\Big(\prod_{\lambda(h)=\lambda(g)=0}x_\lambda\Big)\prod_\lambda\frac{\theta(\frac{x_\lambda}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h)-z)}{\theta(\frac{x_\lambda}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h))}e^{2\pi\mathrm{i}z\lambda(h)}[X^{g,h}]$$

which proves the first part of the theorem.

To verify the modular property, we denote

$$\Phi(g,h,\lambda,z,\tau,x):=\frac{\theta(\frac{x}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h)-z)}{\theta(\frac{x}{2\pi\mathrm{i}}+\lambda(g)-\tau\lambda(h))}e^{2\pi\mathrm{i}z\lambda(h)}$$

where λ is a character of the subgroup of G generated by g and h. Then

$$E_{orb}(z,\tau)=\frac{1}{|G|}\sum_{gh=hg}\Big(\prod_{\lambda(g)=\lambda(h)=0}x_\lambda\Big)\prod_\lambda\Phi(g,h,z,\tau,x_\lambda)[X^{g,h}]\tag{4.2}$$

where we suppress (X,G) from the notations for the sake of brevity. We have:

$$\Phi(g,h,\lambda,z+1,\tau,x)=-\mathrm{e}^{2\pi\mathrm{i}\lambda(h)}\cdot\Phi(g,h,\lambda,z,\tau,x)$$

and hence $Ell_{orb}(z+n,\tau)=(-1)^{dn}Ell_{orb}(z,\tau)$, since by assumption $n\cdot\sum\lambda(h)\in\mathbf{Z}$. It is clear that

$$\Phi(g,h,\lambda,z,\tau+1,x)=\Phi(gh^{-1},h,\lambda,z,\tau,x)$$

and hence $Ell_{orb}(z,\tau+1)=Ell_{orb}(z,\tau)$. We have:

$$\Phi(g,h,\lambda,z+n\tau,\tau,x)=(-1)^n\mathrm{e}^{-2\pi\mathrm{i}nz-\pi\mathrm{i}n^2\tau}\mathrm{e}^{nx+2\pi\mathrm{i}n\lambda(g)}\cdot\Phi(g,h,\lambda,z,\tau,x)$$

and hence

$$Ell_{orb}(z+n\tau,\tau)=(-1)^{dn}e^{-2\pi\mathrm{i}dnz-\pi\mathrm{i}dn^2\tau}Ell_{orb}(z,\tau)$$

since X is Calabi-Yau and $n\lambda(g)\in\mathbf{Z}$. Finally,

$$\Phi(g,h,\lambda,\frac{z}{\tau},\ -\frac{1}{\tau},\frac{x}{\tau})=\frac{\theta(-\frac{z}{\tau}+\frac{x}{2\pi i\tau}+\lambda(g)+\frac{\lambda(h)}{\tau},-\frac{1}{\tau})}{\theta(\frac{x_\lambda}{2\pi\mathrm{i}}+\lambda(g)+\frac{\lambda(h)}{\tau},-\frac{1}{\tau})}e^{\frac{2\pi\mathrm{i}z\lambda(h)}{\tau}}=$$

$$e^{\frac{\pi\mathrm{i}z^2}{\tau}-\frac{2\pi\mathrm{i}z}{\tau}(\frac{x}{2\pi\mathrm{i}}+\lambda(g)\tau+\lambda(h))}\frac{\theta(-z+\frac{x}{2\pi\mathrm{i}}+\lambda(g)\tau+\lambda(h),\tau)}{\theta(\frac{x}{2\pi\mathrm{i}}+\lambda(g)\tau+\lambda(h),\tau)}e^{\frac{2\pi\mathrm{i}z\lambda(h)}{\tau}}=$$

$$e^{\frac{\pi\mathrm{i}z^2}{\tau}-\frac{zx}{\tau}}\cdot\frac{\theta(-z+\frac{x}{2\pi\mathrm{i}}+\lambda(g)\tau+\lambda(h),\tau)}{\theta(\frac{x}{2\pi\mathrm{i}}+\lambda(g)\tau+\lambda(h),\tau)}e^{2\pi\mathrm{i}z(-\lambda(g))}=e^{\frac{\pi\mathrm{i}z^2}{\tau}-\frac{zx}{\tau}}\cdot\Phi(h,g^{-1},\lambda,z,\tau,x).$$

Then the Jacobi transformation properties follows easily from (4.2), similar to the proof of Theorem 2.2 in [12].

It is straightforward to see from (4.2) that orbifold elliptic genus is holomorphic and has the Fourier expansion with non-negative powers of q. □

We will apply our definition of the orbifold elliptic genus to symmetric products of a smooth variety. This will give a mathematical justification of the physical calculation performed in [17]. More precisely, we calculate the generating function for the orbifold elliptic genera introduced above for the action of the symmetric groups. Our calculation to certain extent follows [17], but we now have precise mathematical definitions.

Theorem 4.4. *Let X be a smooth variety with elliptic genus $\sum_{m,l} c(m,l) y^l q^m$, where elliptic genus is normalized as in* [17] *and* [12]*. Then*

$$\sum_{n\geq 0} p^n Ell_{orb}(X^n, \Sigma_n; y, q) = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1-p^i y^l q^m)^{c(mi,l)}}.$$

We shall start with the following lemma essentially contained in ([17], Section 2.2), which we include only for completeness.

Lemma 4.5. *Let $V = V_{\text{even}} \oplus V_{\text{odd}}$ be a supersymmetric space and A and B be two commuting operators preserving parity decomposition of V, such that B has only non-negative integer eigenvalues. We assume that V splits into a direct sum of eigenspaces V_m of the operator B, and each V_m is finite-dimensional. Define*

$$\chi(V)(y,q) = \mathrm{Supertrace}_V y^A q^B := \mathrm{tr}_{V_{\text{even}}}(y^A q^B) - \mathrm{tr}_{V_{\text{odd}}}(y^A q^B) = \sum_{m,l} d(m,l) q^m y^l$$

where $d(m,l)$ is the superdimension of the space $V_{m,l} = \{v \in V | Av = lv, Bv = mv\}$. The operators A and B act on the space of invariants of the symmetric group acting on $V^{\otimes^N}$ and

$$\sum_N p^N \mathrm{Supertrace}_{Sym^N(V)} y^A q^B = \prod_{m,l} \frac{1}{(1-pq^m y^l)^{d(m,l)}}$$

where the right hand-side is expanded as a power series in q and p.

Proof. It is easy to see that it is enough to check the lemma for a one-dimensional space $V = V_{m,l}$. If V is even, then

$$\sum_N p^N \mathrm{Supertrace}_{Sym^N(V)} y^A q^B = \sum_{N\geq 0} p^N y^{Nl} q^{Nm} = (1-pq^m y^l)^{-\mathrm{superdim} V}.$$

If V is odd, then

$$\sum_N p^N \mathrm{Supertrace}_{Sym^N(V)} y^A q^B = 1 - p y^l q^m = (1-pq^m y^l)^{-\mathrm{superdim} V}.$$

□

We are now ready to prove Theorem 4.4.

Proof. We observe that for a fixed k the conjugacy classes of Σ_k are indexed by the numbers a_i of cycles of length i in the permutation. For each $h \in \Sigma_k$ the fixed point set $(X^k)^h$ consists of the Cartesian products of several copies of X, one for each cycle. For a cycle of length i the corresponding X is embedded into X^i. The

centralizer group is a semidirect product of its normal subgroup $\prod_i (\mathbf{Z}/i\mathbf{Z})^{a_i}$ which acts by cyclic permutations inside cycles of h and the product of symmetric groups $\prod_i \Sigma_{a_i}$ that act by permuting cycles of the same length.

Our definition of the elliptic genus then gives

$$\sum_{n\geq 0} p^n Ell_{orb}(X^n,\Sigma_n;y,q) = \sum_{a_1,a_2,\dots,a_n} p^{a_1+2a_2+\ldots+na_n} y^{-\frac{\dim X}{2}(a_1+2a_2+\ldots+na_n)}$$

$$\times \prod_{i=1}^n y^{a_i F(i-cycle,X\subseteq X^i)} \chi\Big(\big(H^\bullet(V_{i-cycle,X\subseteq X^i})^{\otimes a_i}\big)^{\Sigma_{a_i}\rtimes(\mathbf{Z}/i\mathbf{Z})^{a_i}}\Big) \tag{4.3}$$

$$= \prod_{i=1}^\infty \chi\Big(Sym^\bullet\big(p^i y^{-i\dim X/2+F(i-cycle,X\subseteq X^i)} H^\bullet(V_{i-cycle,X\subseteq X^i})^{\mathbf{Z}/i\mathbf{Z}}\big)\Big).$$

The symbol Sym here should be interpreted as the supersymmetric product where the cohomology of $V_{h,X^h\subseteq X}$ is given parity by the sum of the cohomology number and the parity of the exterior algebras.

We will now calculate

$$\chi_i(y,q) = \chi\big(p^i y^{-i\dim X/2+F(i-cycle,X\subseteq X^i)} H^\bullet(V_{i-cycle,X\subseteq X^i})^{\mathbf{Z}/i\mathbf{Z}}\big).$$

We denote the i-cycle by h and observe that

$$TX^i|_X = \oplus_{j=0,\dots,i-1;\lambda(h)=\frac{j}{i}} TX_j.$$

This implies $F(h,X\subseteq X^i) = \dim X \sum_{j=0}^{i-1}\frac{j}{i} = \dim X\frac{(i-1)}{2}$, which allows us to write

$$\chi_i(y,q) = p^i y^{-\dim X/2}\chi\Big[\big[H^\bullet(\otimes_{k\geq 1}\big[(\Lambda^\bullet T^* y q^{k-1})\otimes(\Lambda^\bullet T y^{-1}q^k)\otimes(Sym^\bullet T^* q^k)$$

$$\otimes(Sym^\bullet T q^k)\otimes\big[\otimes_{j=1,\dots,i-1}(\Lambda^\bullet T^* y q^{k-1+\frac{j}{i}})\otimes(\Lambda^\bullet T y^{-1} q^{k-\frac{j}{i}})\otimes(Sym^\bullet T^* q^{k-1+\frac{j}{i}})$$

$$\otimes(Sym^\bullet T q^{k-\frac{j}{i}})\big]\big]\big]^{\mathbf{Z}/i\mathbf{Z}}\Big]$$

$$= p^i y^{-\dim X/2}\frac{1}{i}\sum_{r=0}^{i-1}\int_X \prod_{l=1}^{\dim X} x_l \prod_{k\geq 1}\prod_{m=0}^{i-1}\frac{(1-yq^{k-1+\frac{m}{i}}\xi^{mr}\mathrm{e}^{-x_l})(1-y^{-1}q^{k-\frac{m}{i}}\xi^{-mr}\mathrm{e}^{x_l})}{(1-q^{k-1+\frac{m}{i}}\xi^{mr}\mathrm{e}^{-x_l})(1-q^{k-\frac{m}{i}}\xi^{-mr}\mathrm{e}^{x_l})}$$

$$= p^i y^{-\dim X/2}\frac{1}{i}\sum_{r=0}^{i-1}\int_X \prod_{l=1}^{\dim X} x_l \prod_{j\geq 1}\frac{(1-yq^{\frac{j-1}{i}}\xi^{(j-1)r}\mathrm{e}^{-x_l})(1-y^{-1}q^{\frac{j}{i}}\xi^{jr}\mathrm{e}^{x_l})}{(1-q^{\frac{j-1}{i}}\xi^{(j-1)r}\mathrm{e}^{-x_l})(1-q^{\frac{j}{i}}\xi^{jr}\mathrm{e}^{x_l})}$$

$$= p^i\frac{1}{i}\sum_{r=0}^{i-1} Ell(X;y,q^{\frac{1}{i}}\xi^r) = \sum_{m,l} c(mi,l)y^l q^m.$$

Here we have denoted the primitive i-th root of unity by ξ. Now Lemma 4.5 finishes the proof. □

Remark 4.6. In [36] the authors conjectured an equivariant version of 4.4. Its proof follows using the same arguments as above. More precisely, we have the following. Let X and G be as above and let $G\wr\Sigma_n$ be the wreath product (consisting of pairs $((g_1,...,g_n);\sigma), g_i\in G, \sigma\in\Sigma_n$ with multiplication: $((g_1,...,g_n);\sigma_1)\cdot$

$((h_1,...,h_n);\sigma_2) = ((g_1 \cdot h_{\sigma_1^{-1}(1)},...,g_n \cdot h_{\sigma_1^{-1}(n)});\sigma_1\sigma_2))$. $G \wr \Sigma_n$ acts in an obvious way on X^n and if $Ell_{orb}(X,G;y,q) = \sum c_G(m,l)y^l q^m$ then

$$\sum_{n\geq 0} p^n Ell_{orb}(X^n, G\wr\Sigma_n; y,q) = \prod_{i=1}^{\infty}\prod_{l,m}\frac{1}{(1-p^i y^l q^m)^{c_G(mi,l)}}. \tag{4.4}$$

To obtain a proof of this formula, one should make the following changes in the above proof of Theorem 4.4. Using the description of the conjugacy classes in wreath products (cf. for example [27]) $\sum_{n\geq 0} p^n Ell_{orb}(X^n, G\wr\Sigma_n; y, q)$ can be rewritten as the right hand side of the first row of (4.3) with summation taken over collections $\{h\}, a_1, ..., a_n$ where a_i as earlier are positive integers and $\{h\}$ runs through all conjugacy classes in G. The same transformation as was used in (4.3) now yields the product over i and $\{h\}$ of terms in which invariants are taken for the semidirect product of the centralizer of h and $\mathbf{Z}/i\mathbf{Z}$ with the sheaf V constructed for X^h. Finally, each term in this product is the graded dimension of a supersymmetric algebra, which Lemma 4.5 expresses in terms of $\chi_{i,\{h\}}$. Calculation similar to the above calculation of χ_i identifies $\chi_{i,\{h\}}$ with $\sum_{m,l} c_{\{h\}}(mi,l)y^l q^m = y^{-dim X/2+F(h,X^h\subseteq X)}\frac{1}{|C(h)|}\sum_{g\in C(h)} L(g, V_{h,X^h\subseteq X})$ (the component of the orbifold elliptic genus corresponding to the conjugacy class $\{h\}$). This yields (4.4).

5. Comparison of different notions of elliptic genera

It is natural to ask how the orbifold elliptic genus of X/G is related to its singular elliptic genus. To begin with, even in the case $|G| = 1$, these two genera differ by a normalization factor. In addition, when $\mu : X \to X/G$ has a ramification divisor $D = \sum_i(\nu_i - 1)D_i$, one has to compare $Ell_{orb}(X,G;y,q)$ not to $\widehat{Ell}(X/G;y,q)$ but rather to $\widehat{Ell}(X/G,\Delta_{X/G};y,q)$ where

$$\Delta_{X/G} := \sum_j \left(\frac{\nu_j - 1}{\nu_j}\right)\mu(D_j)$$

with the sum taken among representatives D_j of the orbits of the action of G on the components of the ramification divisor.

Conjecture 5.1. Let X be a smooth algebraic variety equipped with an effective action of a finite group G. Then

$$Ell_{orb}(X,G;y,q) = \left(\frac{2\pi\mathrm{i}\,\theta(-z,\tau)}{\theta'(0,\tau)}\right)^{\dim X}\widehat{Ell}(X/G,\Delta_{X/G};y,q)$$

where $\Delta_{X/G}$ is defined above.

We will now present some evidence to support this conjecture.

Proposition 5.2. *Conjecture 5.1 holds in the limit $\tau \to \mathrm{i}\infty$.*

Proof. At $q = 0$ the function Ell_{orb} specializes to $E_{orb}(y,1)$ of [4]. Then the result of [15] allows one to rewrite it in terms of $E_{st}(y,1)$, and Proposition 3.9 finishes the proof. □

Proposition 5.3. *Conjecture 5.1 holds in the case when X is a smooth toric variety and G is a subgroup of the big torus of X.*

Proof. Let Σ be the defining cone of X in the lattice N, see for example [14]. Let n_i be the generators of one-dimensional cones of Σ. The group G can be identified with N'/N where N' is a suplattice of N of finite coindex. Then the variety X/G is given by the same cone Σ in the new lattice N'. The map $\mu : X \to X/G$ has ramification if and only if for some one-dimensional rays of Σ points n_i are no longer minimal in the new lattice.

Torus-invariant divisors on a toric variety correspond to piecewise linear functions on the fan. It is easy to see that the definition of $\Delta_{X/G}$ assures that the piece-wise linear linear function that takes values (-1) on all n_i gives the divisor $K_{X/G} + \Delta_{X/G}$. We denote this piece-wise linear function by deg. One can show that

$$Ell_{orb}(X, G; y, q) = \left(\frac{2\pi \mathrm{i}\theta(-z,\tau)}{\theta'(0,\tau)}\right)^{\dim X} f_{N',\deg z}(q)$$

where $f_{N',\deg z}(q)$ is the function defined in [11]. More explicitly,

$$f_{N',\deg z}(q) = \sum_{m\in(N')^*}\Big(\sum_{C\in\Sigma}(-1)^{\mathrm{codim} C}\mathrm{a.c.}\sum_{n\in C\cap N'} q^{m\cdot n}\mathrm{e}^{2\pi \mathrm{i} z\deg(n)}\Big),$$

where *a.c.* means analytic continuation. The proof of this fact is based on the explicit calculation of the Euler characteristics of the bundles $V_{X^h\subseteq X}$ by means of Čech cohomology. The calculation is very similar to that of Theorem 3.4 of [11] and is left to the reader. We remark that the sum over h in Definition 4.1 facilitates the change from N to N', while taking $C(h)$-invariants is responsible for the switch from N^* to its sublattice $(N')^*$.

Now let $Y \to X/G$ be a toric desingularization of X/G given by the subdivision Σ_1 of Σ. We denote the codimension one strata of Y by E_k, and the generators of the corresponding one-dimensional cones of Σ_1 by r_k. We also denote the first Chern classes of the corresponding divisors by e_k and get

$$\widehat{Ell}(X/G, \Delta_{X/G}; y, q) = \int_Y \Big(\prod_l \frac{(\frac{y_l}{2\pi \mathrm{i}})\theta(\frac{y_l}{2\pi \mathrm{i}} - z)\theta'(0)}{\theta(-z)\theta(\frac{y_l}{2\pi \mathrm{i}})}\Big)\times\Big(\prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} - (\alpha_k+1)z)\theta(-z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} - z)\theta(-(\alpha_k+1)z)}\Big)$$

where $c(TY) = \prod_l(1 + y_l)$ and $\alpha_k = \deg(r_k) - 1$. We use $c(TY) = \prod_k(1 + e_k)$ to rewrite $\widehat{Ell}(X/G, \Delta_{X/G}; y, q)$ as

$$\int_Y \Big(\prod_k \frac{(\frac{e_k}{2\pi \mathrm{i}})\theta(\frac{e_k}{2\pi \mathrm{i}} - \deg(r_k)z)\theta'(0)}{\theta(-\deg(r_k)z)\theta(\frac{e_k}{2\pi \mathrm{i}})}\Big)$$

which equals $f_{N',\deg z}(q)$ by Theorem 3.4 of [11]. We have used here the fact that f does not change when the fan is subdivided. □

Remark 5.4. The above proposition was the main motivation behind our definition of the singular elliptic genus.

Proposition 5.5. *Conjecture 5.1 holds for* $\dim X = 1$.

Proof. Expanding θ functions as (linear) polynomials in cohomology classes, one obtains that singular genus is equal to $(2g-2)\theta'(-z)/(2\pi \mathrm{i}\theta(-z))$ plus sum of contributions of singular points that depend on the ramification numbers only. Here g is the genus of X/G. For the orbifold genus, one needs to notice that $h = \mathbf{id}$ term gives $(2g-2)\theta'(-z)/(2\pi \mathrm{i}\theta(-z))$ plus contributions of points, because it is the Euler characteristics of the bundle on the quotient that is the usual elliptic genus bundle

twisted at the ramification points. Since the equality holds in the toric case of the d-fold covering of $\mathbf{P}^1$ by $\mathbf{P}^1$, which has two points of ramification $(d-1)$, the extra terms for two genera coincide, which finishes the proof. $\square$

One would also want to compare singular elliptic genus to the elliptic genus defined for toric varieties and Calabi-Yau hypersurfaces in toric varieties in [12]. It turns out that these definitions agree, up to a normalization. We will explain the Calabi-Yau case in more detail, and will leave the toric case to the reader. We need to recall the combinatorial description of Calabi-Yau hypersurfaces in toric varieties and the previous definition of their elliptic genera.

Let M_1 and N_1 be dual free abelian groups of rank $d+1$. Denote by M and N the dual free abelian groups $M = M_1 \oplus \mathbf{Z}$ and $N = N_1 \oplus \mathbf{Z}$. Element $(0,1) \in M$ is denoted by deg and element $(0,1) \in N$ is denoted by $\deg^*$. There are dual reflexive polytopes $\Delta \in M_1$ and $\Delta^* \in N_1$ which give rise to dual cones $K \subset M$ and $K^* \subset N$. Namely, K is a cone over $(\Delta, 1)$ with vertex at $(0,0)_M$, and similarly for K^*. There is a complete fan Σ_1 on N_1 whose one-dimensional cones are generated by some lattice points in Δ^* (in particular, by all vertices). This fan induces the decomposition of the cone K^* into subcones, each of which includes $\deg^*$. Let us denote this decomposition by Σ. A generic Calabi-Yau hypersurface X_f of the family given by the above combinatorial data is determined by a choice of coefficients f_m for each $m \in (\Delta, 1)$.

Elliptic genus of X_f was defined in [12] as the graded Euler characteristic of a certain sheaf of vertex algebras on X_f. We will not need to recall the definition of this sheaf in view of the following combinatorial formula for the elliptic genus:

Proposition 5.6. *The elliptic genus $Ell(X_f; y, q)$ of the Calabi-Yau hypersurface X_f as defined in* [12] *is given by*

$$Ell(X_f; y, q) = y^{-d/2} \sum_{m \in M} a.c. \left(\sum_{n \in K^*} y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n + m\cdot\deg^*} G(y,q)^{d+2} \right),$$

where a.c. stands for analytic continuation and

$$G(y,q) = \prod_{k\geq 1} \frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2}.$$

Proof. Combine Proposition 4.2 and Definition 5.1 of [12]. $\square$

Theorem 5.7. *Elliptic genus of the Calabi-Yau hypersurface X_f of dimension d defined above and its singular elliptic genus are related by the formula*

$$Ell(X_f; y, q) = \left(\frac{2\pi \mathrm{i}\theta(-z,\tau)}{\theta'(0,\tau)} \right)^d \widehat{Ell}(X_f; y, q).$$

Proof. First of all, observe that

$$y^{-1/2} G(y,q) = \frac{2\pi \mathrm{i}\theta(-z,\tau)}{\theta'(0,\tau)},$$

due to the product formulas for $\theta(z,\tau)$ and $\theta'(0,\tau)$, see [13]. Therefore, we only need to show that

$$\widehat{Ell}(X_f; y, q) = \sum_{m \in M} a.c. \left(\sum_{n \in K^*} y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n + m\cdot\deg^*} G(y,q)^2 \right).$$

Denote by $\deg_1$ the piece-wise linear function on N_1 whose value on the generators of the one-dimensional cones of Σ_1 is 1. Notice that K^* consists of all points $(n_1, l) \in N$ such that $l \geq \deg_1(l)$. In addition, one can replace $\sum_{n\in K}\ldots$ by $\sum_{C\in\Sigma}(-1)^{\mathrm{codim}\Sigma}\ldots$ to get

$$\sum_{m\in M} a.c.\left(\sum_{n\in K^*} y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n + m\cdot\deg^*} G(y,q)^2\right)$$

$$= \sum_{k\in\mathbf{Z}}\sum_{m_1\in M}\sum_{C_1\in\Sigma_1}(-1)^{\mathrm{codim}C_1} a.c. \sum_{n_1\in C_1}\sum_{l\geq\deg_1(n_1)} y^{l-k} q^{m_1\cdot n_1 + lk + k} G(y,q)^2$$

$$= \sum_{k\in\mathbf{Z}}\sum_{m_1\in M}\sum_{C_1\in\Sigma_1}(-1)^{\mathrm{codim}C_1} a.c. \sum_{n_1\in C_1}\sum_{l\geq\deg_1(n_1)} y^{\deg_1(n_1)-k} q^{m_1\cdot n_1 + \deg_1(n_1)k + k}(1-yq^k)^{-1} G(y,q)^2$$

$$= G(y,q)^2 \sum_{k\in\mathbf{Z}} \frac{y^{-k}q^k}{(1-yq^k)} f_{N_1,\deg_1 z}(yq^k, q).$$

Let Σ_1' be a refinement of the fan Σ_1 in N_1 such that the corresponding toric variety $\mathbf{P}_{\Sigma_1'}$ is smooth. Coefficients f_m define a hypersurface X_f' in $\mathbf{P}_{\Sigma_1'}$ which is a resolution of singularities X_f. We denote the codimension one strata of $\mathbf{P}_{\Sigma_1'}$ by D_j, their first Chern classes by d_j and the corresponding generators of one-dimensional cones of Σ_1' by n_j. By Theorem 3.4 of [11], we get

$$G(y,q)^2 \sum_{k\in\mathbf{Z}} \frac{y^{-k}q^k}{(1-yq^k)} f_{N_1,\deg_1 z}(yq^k, q)$$

$$= G(y,q)^2 \sum_{k\in\mathbf{Z}} \frac{y^{-k}q^k}{(1-yq^k)} \int_{\mathbf{P}_{\Sigma_1'}} \prod_j \frac{(\frac{d_j}{2\pi\mathrm{i}})\theta(\frac{d_j}{2\pi\mathrm{i}} - \deg_1(n_j)(z+k\tau))\theta'(0)}{\theta(-\deg_1(n_j)(z+k\tau))\theta(\frac{n_j}{2\pi\mathrm{i}})}$$

$$= \int_{\mathbf{P}_{\Sigma_1'}} \prod_j \frac{(\frac{d_j}{2\pi\mathrm{i}})\theta(\frac{d_j}{2\pi\mathrm{i}} - \deg_1(n_j)z)\theta'(0)}{\theta(-\deg_1(n_j)z)\theta(\frac{n_j}{2\pi\mathrm{i}})} \Big(\sum_{k\in\mathbf{Z}} G(y,q)^2 \frac{y^{-k}q^k}{(1-yq^k)} \mathrm{e}^{k\sum_j d_j\deg_1(n_j)}\Big)$$

$$= \int_{\mathbf{P}_{\Sigma_1'}} \prod_j \frac{(\frac{d_j}{2\pi\mathrm{i}})\theta(\frac{d_j}{2\pi\mathrm{i}} - \deg_1(n_j)z)\theta'(0)}{\theta(-\deg_1(n_j)z)\theta(\frac{n_j}{2\pi\mathrm{i}})} \Big(\sum_{k\in\mathbf{Z}} G(y,q)^2 \frac{y^{-k}q^k}{(1-yq^k)} \mathrm{e}^{k\sum_j d_j\deg_1(n_j)}\Big)$$

We denote $D = \sum_j \deg_1(n_1)D_j$ and $d = c_1(D)$. Because of Proposition 3.2 of [12], we get

$$\sum_{k\in\mathbf{Z}} G(y,q)^2 \frac{y^{-k}q^k}{(1-yq^k)} \mathrm{e}^{k\sum_j d_j\deg_1(n_j)} = \frac{G(e^d q, q)G(y,q)}{G(y^{-1}e^d q, q)} = \frac{2\pi\mathrm{i}\theta(\frac{d}{2\pi\mathrm{i}})\theta(-z,\tau)}{\theta(\frac{d}{2\pi\mathrm{i}} - z)\theta'(0)}$$

which gives

$$\sum_{m\in M} a.c.\left(\sum_{n\in K^*} y^{n\cdot\deg - m\cdot\deg^*} q^{m\cdot n + m\cdot\deg^*} G(y,q)^2\right)$$

$$= \int_{\mathbf{P}_{\Sigma_1'}} \prod_j \frac{(\frac{d_j}{2\pi\mathrm{i}})\theta(\frac{d_j}{2\pi\mathrm{i}} - \deg_1(n_j)z)\theta'(0)}{\theta(-\deg_1(n_j)z)\theta(\frac{n_j}{2\pi\mathrm{i}})} \Big(\frac{2\pi\mathrm{i}\theta(\frac{d}{2\pi\mathrm{i}})\theta(-z,\tau)}{\theta(\frac{d}{2\pi\mathrm{i}} - z)\theta'(0)}\Big)$$

Observe now that $D = \pi^*(-K_{\mathbf{P}_{\Sigma_1}})$ where $\pi : \mathbf{P}_{\Sigma_1'} \to \mathbf{P}_{\Sigma_1}$ is the resolution induced by the subdivision of the fan. In addition, X_f is a zero set of a section of D. Hence, the adjunction formula gives

$$c(T_{X_f'}) = i^* c(T\mathbf{P}_{\Sigma_1'})/(1 + i^* d)$$

where $i : X'_f \to \mathbf{P}_{\Sigma'_1}$ is the embedding. The exceptional divisors of $X'_f \to X_f$ are $D_j \cap X'_f$ (unless $\dim \pi(D_j) = 0$), and their discrepancies are equal to $\deg(n_j) - 1$. Then it is easy to see that the above expression is precisely the singular elliptic genus of X_f. ∎

Remark 5.8. The case of toric varieties is a straightforward application of Theorem 3.4 of [11] and is left to the reader.

Remark 5.9. The above calculations indicate that for any smooth variety $\mathbf{P}$ of dimension $d+1$ one can define a weak Jacobi form of weight d and index 0 which coincides with the singular elliptic genus of the Calabi-Yau hypersurface in $\mathbf{P}$ if $\mathbf{P}$ has smooth anticanonical divisors. Otherwise, the formula gives elliptic genus of "virtual" Calabi-Yau hypersurface in $\mathbf{P}$. One can also interpret this Jacobi form as an elliptic genus of $(d+1,1)$-dimensional Calabi-Yau supermanifold $\Pi K X$ (canonical line bundle over X, considered as an odd bundle).

6. Cobordism invariance of orbifold elliptic genus

We shall view $Ell_{sing}(X/G)$ and $Ell_{orb}(X, G)$ as invariants of G-action on X and will work in the category of stably almost complex manifolds.

Lemma 6.1. *Singular elliptic genus is an invariant of complex G-cobordism.*

Proof. We shall consider cobordisms of pairs (X, D) ([35]) where X is a stably almost complex manifold (i.e. C^∞ manifold such that a direct sum of a trivial bundle ϵ with the differentiable tangent bundle T_X admits a complex structure) and $D = \cup D_i$ is a finite union of codimension one stably almost complex submanifolds (i.e. $T_{D_i} \oplus \epsilon$ is a complex subbundle in $\epsilon \oplus T_X$) satisfying the following normal crossing condition: at each point of $D_{i_1} \cap ... \cap D_{i_k}$ the union of (stabilized by adding trivial bundles) tangent spaces $T_{D_{i_j}} \oplus \epsilon$ is given in the (stabilized) tangent space to X by $l_1 \cdots l_k = 0$ where l_i are linearly independent complex linear forms. A pair (X, D) is cobordant to zero if there exist a C^∞-manifold Y with a complex structure on the stable tangent bundle and a system of submanifolds $\cup E_i$ such that $\partial Y = X$ and $\cup \partial E_i = \cup D_i$. As usual, the disjoint union and product provide the ring structure on cobordism classes.

Notice that the numbers $c_{i_1} \cup ... \cup c_{i_k} \cup [D_{j_1}] \cup ... \cup [D_{j_k}]([X])$ where $[D_i]$ are the classes in $H^2(X, \mathbf{Z})$ dual to submanifolds D_i, $[X] \in H_{2\dim_{\mathbf{C}}\ X}$ is the fundamental class of X and $\sum_j i_j + k = \dim_{\mathbf{C}}\ X$ are invariants of cobordism of such pairs (indeed, if $X = \partial Y$ and $j : X \to Y$ then this number is $j^*(c_{i_1} \cup ... \cup c_{i_k} \cup [E_{j_1}] \cup ... \cup [E_{j_k}])([X]) = c_{i_1} \cup ... \cup c_{i_k} \cup [D_{j_1}] \cup ... \cup [D_{j_k}](j_*[X]) = 0$ since X is homologous to zero in Y). The lemma therefore will follow if we shall show that for an almost complex null-cobordant G-manifold X the quotient X/G admits a resolution of singularities $(\widetilde{X/G}, D)$, where $D = \cup D_i$ is the exceptional locus, such that this pair is cobordant to zero.

If $X = \partial Y$, where Y is a G-manifold, we can construct resolution of Y/G as follows. Let H be a subgroup of G and $Y_H = \{y \in Y | Stab\ y = H\}$. Then Y_H are smooth submanifolds of Y (possibly with boundary) providing a stratification of Y. Let $C(H)$ be the union of subgroups of G conjugate to H. Then $Y_{C(H)} = \cup_{H' \in C(H)} Y_{H'}$ is still a submanifold of Y and the group G acts on $Y_{C(H)}$ so that $Y_{C(H)} \to Y_{C(H)}/G$ is an unramified cover (of degree $[G : H]$). In particular, $Y_{C(H)}/G$ is a smooth manifold and these manifolds for all $H \subset G$ provide a

stratification of Y/G such that Y/G is equisingular along each stratum $Y_{C(H)}/G$. A small regular neighborhood of each stratum in Y/G is isomorphic to a bundle ξ_H over $Y_{C(H)}/G$ with the fiber isomorphic to V/H where V is a fiber of the normal bundle to Y_H in Y over a point of Y_H (this presentation is independent of a point in Y_H and representations at points of Y_H and $Y_{H'}$ are isomorphic for conjugate H and H').

For each quotient singularity V/H let us fix the universal desingularization constructed by Bierstone-Milman (cf. Theorem 13.2 of [8]). Its universality assures that it is equivariant with respect to the centralizer of H in $GL(V)$. Hence one can use the transition functions of ξ_H to construct the fibration $\tilde{\xi}_H$ with the same base as ξ_H and having as its fiber the universal resolution of V/H. Moreover, due to universality of canonical resolution (cf. Theorem 13.2 in [8]) this property assures that $\tilde{\xi}_H$ corresponding to different classes of conjugate subgroups H can be glued together yielding an almost complex manifold which boundary is the pair $(\widetilde{X/G}, D)$ where D is the exceptional set of the universal resolution of X/G. This proves the lemma. $\square$

Lemma 6.2. *Orbifold elliptic genus is an invariant of G-cobordism.*

Proof. Let X be a null-cobordant G-manifold. Then for each $g \in G$ the pair $X^g, \nu(X^g, X)$ where $\nu(X^g, X)$ is the normal bundle of the fixed point set X^g in X is cobordant to zero as well. Since the contribution of the term in Ell_{orb} corresponding to a conjugacy class $[g]$ is a combination of the products of Chern classes of X^g and $\nu(X^g, X)$ evaluated on the fundamental class of X^g this contribution is zero. This yields the lemma. $\square$

Corollary 6.3. *Conjecture 5.1 is true for $G = \mathbf{Z}/2\mathbf{Z}$.*

Proof. This follows from the result of Kosniowski ([29]) describing generators of $\mathbf{Z}/p\mathbf{Z}$-cobordisms. If $p = 2$, then additive generators of cobordism group in any dimension are toric varieties with group being a subgroup of the big torus. Hence Proposition 5.3 yields the claim. $\square$

References

[1] D. Abramovich, K. Karu, K. Matsuki, J. Włodarsczyk, *Torification and Factorization of Birational Maps*, preprint math.AG/9904135.

[2] M. Atiyah, I. Singer, *The index of elliptic operators. III*, Ann. of Math. (2), **87** 1968, 546–604.

[3] V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 1–32, World Sci. Publishing, River Edge, NJ, 1998.

[4] V. V. Batyrev, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, J. Eur. Math. Soc. (JEMS),**1**(1999), no. 1, 5–33.

[5] V. V. Batyrev, *Canonical abelianization of finite group actions*, preprint math.AG/0009043.

[6] V, V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, New trends in algebraic geometry (Warwick, 1996), 1–11, London Math. Soc. Lecture Note Ser., **264**, Cambridge Univ. Press, Cambridge, 1999.

[7] V. V. Batyrev, D. I. Dais, *Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry*, Topology, **35** (1996), no. 4, 901-929.

[8] E. Bierstone, P. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math., **128** (1997), no. 2, 207–302.

[9] A. Borel et al., *Intersection cohomology*, Notes on the seminar held at the University of Bern, Bern, 1983. Progress in Mathematics, 50. Swiss Seminars. Birkhäuser Boston, Inc., Boston, Mass., 1984.

[10] L. Borisov, *Vertex Algebras and Mirror Symmetry*, to appear in Comm. Math. Phys., preprint math.AG/9809094.
[11] L. A. Borisov, P. E. Gunnells, *Toric varieties and modular forms*, preprint math.NT/9908138.
[12] L. Borisov, A. Libgober, *Elliptic Genera of Toric Varieties and Applications to Mirror Symmetry*, Invent. Math., **140** (2000), no. 2, 453–485.
[13] K. Chandrasekharan, *Elliptic functions*, Fundamental Principles of Mathematical Sciences, **281**, Springer-Verlag, Berlin-New York, 1985.
[14] V. I. Danilov, *The Geometry of Toric Varieties*, Russian Math. Surveys, **33**(1978), 97-154.
[15] J. Denef, F. Loeser, *Motivic integration, quotient singularities and the McKay correspondence*, preprint math.AG/9903187.
[16] J. Denef, F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232.
[17] R. Dijkgraaf, D. Moore, E. Verlinde, H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Comm. Math. Phys. **185** (1997), no. 1, 197–209.
[18] M. Eichler, D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics, **55**, Birkhuser Boston, Inc., Boston, Mass., 1985.
[19] G. Ellingsrud, L. Gottsche, M. Lehn, *On the Cobordism Class of the Hilbert Scheme of a Surface*, preprint math.AG/990409.
[20] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math., **90** (1968), 511–521.
[21] W. Fulton, *Intersection Theory, second edition*, Springer-Verlag, Berlin, 1998.
[22] L.Göttsche, W.Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann., **296** (1993), no. 2, 235–245.
[23] V. Gritsenko,*Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms*, preprint math.AG/9906190, to appear in Algebra i Analiz (St. Petersburg Math. Journal) 11:5 (1999).
[24] A.Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2), **9** (1957), 119–221.
[25] F. Hirzebruch, *Topological methods in Algebraic Geometry*, translated from German and Appendix One by R. L. E. Schwarzenberger. With a preface to the third English edition by the author and Schwarzenberger. Appendix Two by A. Borel. Reprint of the 1978 edition. Classics in Mathematics, Springer-Verlag, Berlin, 1995.
[26] F.Hirzebruch and T.Höfer *On the Euler number of an orbifold*, Math. Ann. **286** (1990), no. 1-3, 255–260.
[27] A. Kerber, *Representations of the permutation groups*, Lecture Notes in Math, **240**, Springer Verlag, Berlin-New York, 1971.
[28] J. Kollár, Y. Miyaoka, S. Mori, *Higher-dimensional complex geometry*, Astérisque, **166** (1988).
[29] C. Kosniowski, *Generators of the Z/p bordism ring. Serendipity*, Math. Z., **149** (1976), no. 2, 121–130.
[30] I. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Cambridge Philos. Soc. **58** (1962), 563–568.
[31] F. Malikov, V. Schechtman, A. Vaintrob, *Chiral de Rham complex*, Comm. Math. Phys., **204**(1999), no. 2, 439–473.
[32] J. McKay, *Graphs, singularities, and finite groups*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), 183–186, Proc. Sympos. Pure Math., **37**, Amer. Math. Soc., Providence, R.I., 1980.
[33] P. H. Siegel, *Witt spaces: a geometric cycle theory for KO-homology at odd primes*, Amer. J. Math., **105** (1983), no. 5, 1067–1105.
[34] B. Totaro, *Chern numbers of singular varieties and elliptic homology*, Ann. of Math., **151**(2000), no. 2, 757-791.
[35] C. T. C. Wall, *Cobordism of pairs*, Comm.Math.Helv., **35** (1961), 135-145.
[36] W. Wang, J. Zhou, *Orbifold Hodge numbers of the wreath product orbifolds*, preprint math.AG/0005124.
[37] E. Zaslow, *Topological orbifold models and quantum cohomology rings*, Comm. Math. Phys., **156** (1993), no. 2, 301–331.

Department of Mathematics, Columbia University, New York, NY 10027
E-mail address: lborisov@math.columbia.edu

Department of Mathematics, University of Illinois, Chicago, IL 60607
E-mail address: libgober@math.uic.edu